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Pi Network Study

The pi network equation is derived, and an expression is found for the maximum inductance value and for other properties.

The pi network has been used in radio engineering for quite some time, and it might seem difficult to come up with something new in this area. However, that's not the case. This article will derive the pi network equation, explore the area of dual solutions, obtain an expression for the maximum inductance value, and propose a new algorithm for calculating pi network based on the required out-of-band suppression.

1 – The main relationships in a pi network

Figure 1 shows the pi network circuit, which is used for impedance matching between the source resistance R_s and the load resistance R_l. This circuit is widely used in vacuum tube power amplifiers, where it is necessary to match the high output impedance of the tube with a low antenna impedance. In addition to impedance matching, the pi network also acts as a low-pass filter that suppresses harmonics.

The pi network is conveniently analyzed by representing it as two **L**-networks connected in series back-to-back (**Figure 2**). The first **L**-network converts the source resistance R_s to some intermediate resistance R_v — let's call it the virtual resistance. The second **L**-network converts the virtual resistance R_v to the load resistance R_l. In further calculations, we will assume that $R_v < R_l < R_s$.

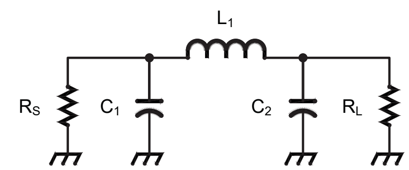
Let's write down the main relationships for the pi network. The Q factors of the **L**-networks are respectively equal to

$$Q_1 = \sqrt{\frac{R_s}{R_v} - 1} \qquad (1)$$

$$Q_2 = \sqrt{\frac{R_l}{R_v} - 1} \qquad (2)$$

The condition $R_v < R_l < R_s$ implies a limitation on the Q factor of the first **L**-network

$$Q_1 > \sqrt{\frac{R_s}{R_l} - 1} \qquad (3)$$



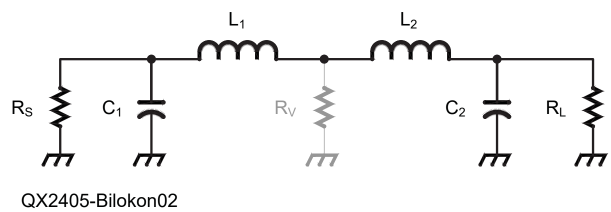


Figure 2 — The pi network is conveniently analyzed as two L-networks connected back-to-back in series.

The reactances of the pi network elements are determined by the following formulas:

$$XC_1 = \frac{R_s}{Q_1},\ XC_2 = \frac{R_l}{Q_2} \qquad (4)$$

$$\begin{aligned} XL &= XL_1 + XL_2 \\ &= R_v Q_1 + R_v Q_2 \\ &= R_v\left(Q_1 + Q_2\right) \end{aligned} \qquad (5)$$

The following relationships can be easily derived from formulas **(1)** and **(2)**:

$$R_v = \frac{R_s}{Q_1^2 + 1} = \frac{R_l}{Q_2^2 + 1} \qquad (6)$$

$$\frac{Q_1^2 + 1}{Q_2^2 + 1} = \frac{R_s}{R_l} \qquad (7)$$

2 – Pi network equation

From **(7)**, it follows that for given source and load resistances, the Q factors of the **L**-network sections are related to each other. Therefore, we can exclude one of the values. Using **(7)**, we obtain

$$Q_2 = \sqrt{\frac{R_l\left(Q_1^{\,2}+1\right)}{R_s} - 1} \qquad (8)$$

Let's derive a formula relating the inductance to the Q factor of the **L**-network. Substituting formulas **(6)** and **(8)** into formula **(5)**, we eliminate Q_2 and obtain the following expression for the inductance:

$$XL = R_v\left(Q_1 + Q_2\right) = \frac{R_s}{Q_1^{\,2}+1}\left(Q_1 + \sqrt{\frac{R_l\left(Q_1^{\,2}+1\right)}{R_s} - 1}\right) \qquad (9)$$

At first glance, it may seem that the value of XL monotonically decreases with increasing Q_1, but this is not the case. If we plot the function **(9)**, it will have the shape shown in **Figure 3**. The minimum value of Q_1 is determined by inequality **(3)**. It can be seen that the function has a maximum, which is reached at a value of Q_1 greater than the minimum. In the vicinity of this maximum, there is an area where there exist two solutions of Q_1 for the same value of inductance. Let us examine this issue in more detail.

Substituting the minimum value of Q_1 from **(3)** into **(9)**, we get:

$$XL > XL_{Q_1=\sqrt{\frac{R_s}{R_l}-1}} = R_l\sqrt{\frac{R_s}{R_l}-1} = \sqrt{R_l\left(R_s-R_l\right)} \qquad (10)$$

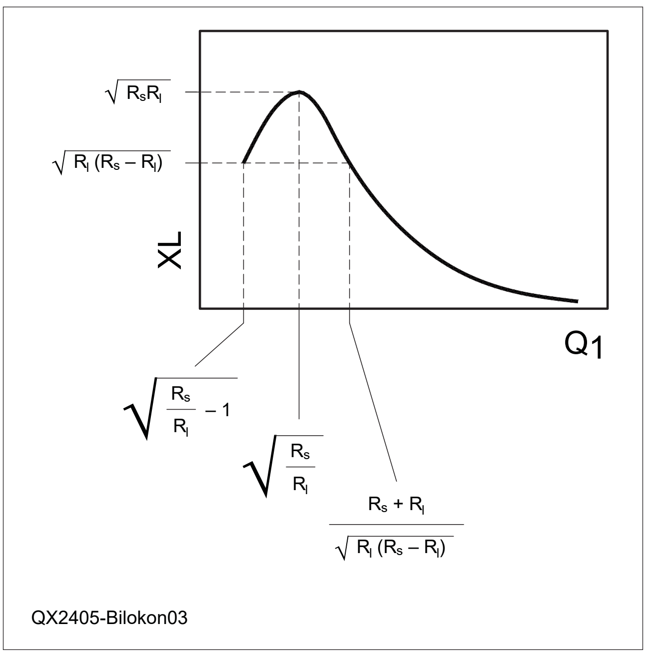


Figure 3 — XL does not decrease monotonically with increasing Q_1.

In the case $XL > \sqrt{R_l\left(R_s-R_l\right)}$ there are two solutions for the pi network values. Let's determine the maximum value of the inductance. We express Q_1 as a function of XL from formula **(9)**. The complete solution is given in **Appendix 1**.

$$Q_1 = \frac{R_s \pm \sqrt{R_s R_l - XL^2}}{XL} \qquad (11)$$

To have a solution, it is necessary for the expression under the square root to be non-negative. That is, the following inequality must hold:

$$R_s R_l - XL^2 \geq 0 \qquad (12)$$

Solving it for XL, we obtain a restriction on the maximum value of inductance in pi network:

$$XL \leq \sqrt{R_s R_l} \qquad (13)$$

A pi network with maximum inductance has quite interesting properties. Capacitors C_1 and C_2 have the same nominal value and their reactance is equal in absolute value to the reactance of the inductor. The value of the virtual resistance R_v is equal to the value of the source and load resistances connected in parallel. Mathematically, this is written quite elegantly:

$$XL = XC_1 = XC_2 = \sqrt{R_s R_l} \qquad (14)$$

$$Q_1 = \sqrt{\frac{R_s}{R_l}},\ Q_2 = \sqrt{\frac{R_l}{R_s}},\ Q_1 Q_2 = 1$$

$$R_v = \frac{R_s R_l}{R_s + R_l}$$

For example, to match 1800 Ω to 50 Ω, the inductance reactance will be $\sqrt{1800 \times 50} = 300$ Ω.

For a frequency of 7.1 MHz, the pi network element values will be: L = 6.725 μH, $C_1 = C_2$ = 74.72 pF. This pi network with maximum inductance will have minimal losses at the operating frequency. However, the out-of-band rejection in it will be one of the worst.

In the range of inductance values $\sqrt{R_l\left(R_s-R_l\right)} < XL < \sqrt{R_l R_s}$, the pi network will have two solutions. For example, for R_s = 1800 Ω, R_l = 50 Ω, XL = 297 Ω, using **(11)** we obtain two values of Q factor: Q_1 = {5.92, 6.2}. For 7.1 MHz, the values of the pi network elements will be as follows: L = 6.66 μH, C_1 = 73.7 pF, C_2 = 16.1 pF, and second solution L = 6.66 μH, C_1 = 77.2 pF, C_2 = 116.7 pF. You can verify this in any circuit simulator.

3 – Revised pi network calculation

The calculation of a pi network is usually performed using the method described in **[1]**. The initial data for the calculation is the input and output resistances, as well as the quality factor of the first **L**-network, which is recommended to be selected in the range of 10 to 15. If we are not satisfied with the calculation result, it is suggested to change the Q factor and perform the calculation again. However, there is no information anywhere about how the choice of Q factor will affect the calculation results. In my opinion, this is the disadvantage of this method.

In articles **[2, 3, 4]**, various attempts were made to build a new mathematical apparatus for describing the pi network. To do this, the author introduces the concept of a generalized quality factor: operating Q of pi network. But as a result, this leads to the same problem of choosing this quality factor depending on the requirements for the designed pi network.

To develop a new pi network calculation algorithm, let's recall what goals it achieves in power amplifi er: (a) impedance matching, (b) fi ltering of unwanted out-of-band emissions. Point (b) is no less important than point (a).

The spectrum from a class-B amplifier with a 90° cutoff angle is described by Berg coefficients **[5]**, which are 0.5 and 0.212 for the 1st and 2nd harmonics, respectively. It is easy to calculate that the level of the second harmonic will be only 7.45 dB below the level of the fundamental signal. At the same time, the level of the third and higher harmonics is much lower than the level of the second harmonic and can be neglected.

One could choose a high Q factor value and hope to achieve high out-of-band suppression. However, increasing Q factor also increases losses and reduces efficiency. Therefore, it is necessary to choose the minimum Q factor value necessary to achieve the required out-of-band suppression.

Thus, the input and output impedance, as well as the level of suppression of the second harmonic, will be the input data for our new pi network calculation algorithm.

Let's use the formula for the suppression of the second harmonic for the **L**-network (see **Appendix 2** for a full derivation).

$$A = 10\ \log_{10}\left(1 + \frac{9Q^4}{4\left(Q^2+1\right)}\right) \tag{15}$$

Here, A is the suppression of the second harmonic in dB, Q is the Q factor of the L-network. Since our pi network contains two **L**-network connected in series, we can write a system of equations:

$$\begin{cases} \left(1 + \frac{9Q_1^4}{4\left(Q_1^2+1\right)}\right)\left(1 + \frac{9Q_2^4}{4\left(Q_2^2+1\right)}\right) = 10^{\frac{A}{10}} \\ \frac{Q_1^{\ 2}+1}{Q_2^{\ 2}+1} = \frac{R_s}{R_l} \end{cases} \tag{16}$$

This is a system of equations with two unknowns, Q_1 and Q_2. It is quite easy to solve numerically on a computer (for example, using spreadsheets). But an analytical solution would be quite cumbersome due to the high degree of equations.

To find an analytical solution, we can simplify **(15)** to the following:

$$A = 10\ \log_{10}\frac{9Q^2}{4} \tag{17}$$

The error of such a simplified formula compared to the original one is no more than 0.1 dB when $Q \geq 5$ and increases to 0.5 dB when $1 \geq Q < 5$. For $Q < 1$, the error increases, and the formula becomes inapplicable.

Pi network contains two series-connected **L**-network, so the simplified attenuation formula for it will look like this:

$$10^{\frac{A}{10}} = \frac{81Q_1^2Q_2^2}{16} \tag{18}$$

Substituting the quality factor values from **(1)** and **(2)** into this equation, we get:

$$10^{\frac{A}{10}} = \frac{81}{16}\left(\frac{R_s}{R_v}-1\right)\left(\frac{R_l}{R_v}-1\right) = \frac{81}{16}\frac{\left(R_s-R_v\right)\left(R_l-R_v\right)}{R_v^2} \tag{19}$$

Let's write this expression as a quadratic equation with respect to R_v

$$\left(K-1\right)R_v^2 + \left(R_s+R_l\right)R_v - R_sR_l = 0 \tag{20}$$

where

$$K = \frac{16}{81}10^{\frac{A}{10}}$$

To simplify, we will consider only one positive root:

$$R_v = \frac{\sqrt{\left(R_s+R_l\right)^2 + 4\left(K-1\right)R_sR_l} - \left(R_s+R_l\right)}{2\left(K-1\right)} \tag{21}$$

Knowing R_v, you can determine the reactance of all pi network elements using **(1)**, **(2)**, and **(5)**. The obtained equation is of sufficient accuracy for practical purposes to allow calculating the pi network with the required out-of-band suppression.

Example of numerical calculation

Let R_s = 1800 Ω, R_l = 50 Ω, and the required suppression of the second harmonic A = 28 dB. It is necessary to calculate the pi network for a frequency of 10 MHz.

$$K = \frac{16}{81}10^{\frac{28}{10}} \cong 124.6$$

$$R_v = \frac{\sqrt{(18000+50)^2 + 4 \times (124.6-1) \times 1800 \times 50} - (1800+50)}{2 \times (124.6-1)} \cong 20.5\ ohms$$

The quality factors of the first and second **L**-network are equal to

$$Q_1 = \sqrt{\frac{R_s}{R_v}-1} = \sqrt{\frac{1800}{20.5}-1} \cong 9.3$$

$$Q_2 = \sqrt{\frac{R_l}{R_v}-1} = \sqrt{\frac{50}{20.5}-1} \cong 1.2$$

Find reactance of elements

$$XC_1 = \frac{R_s}{Q_1} = \frac{1800}{9.3} \cong 193.6\ \Omega$$

$$XC_2 = \frac{R_l}{Q_2} = \frac{50}{1.2} \cong 41.7\ \Omega$$

$$XL = R_v\left(Q_1+Q_2\right) = 20.5\left(9.3+1.2\right) \cong 215.25\ \Omega$$

Find the values of the pi network elements for a frequency of 10 MHz

$$C_1 = \frac{1}{2\pi f X C_1} \cong 82.2 \text{ pF}$$

$$C_2 = \frac{1}{2\pi f X C_2} \cong 382 \text{ pF}$$

$$L = \frac{XL}{2\pi f} \cong 3.43 \ \mu\text{H}$$

Testing in the simulator gives a suppression of the second harmonic equal to 28.16 dB, which corresponds quite accurately to the required value.

Thus **(16)** and **(21)** have been obtained, which allow calculating the parameters of a pi network depending on the required out-of-band suppression.

Appendix 1. Solving the pi network equation

Given equation **(9)**

$$XL = \frac{R_s}{Q_1^{\,2}+1}\left(Q_1 + \sqrt{\frac{R_l\left(Q_1^{\,2}+1\right)}{R_s} - 1}\right)$$

$$\sqrt{\frac{R_l\left(Q_1^{\,2}+1\right)}{R_s} - 1} = \frac{XL\left(Q_1^{\,2}+1\right)}{R_s} - Q_1$$

Square both sides of the equation, move the −1 to the right side and reduce the equation by $Q_1^{\,2}+1$ to get:

$$\frac{R_l}{R_s} = \frac{XL^2\left(Q_1^{\,2}+1\right)}{R_s^{\,2}} - 2Q_1\frac{XL}{R_s} + 1$$

Further, we simplify and transform it into the form of a quadratic equation for Q

$$Q_1^{\,2}\frac{XL^2}{R_s^{\,2}} - 2Q_1\frac{XL}{R_s} + \frac{XL^2}{R_s^{\,2}} - \frac{R_l}{R_s} + 1 = 0$$

Multiply by $\frac{R_s^{\,2}}{XL^2}$

$$Q_1^{\,2} - 2Q_1\frac{R_s}{XL} + 1 - \frac{R_sR_l}{XL^2} + \frac{R_s^{\,2}}{XL^2} = 0$$

Find the discriminant Δ of the equation

$$\Delta = 4\frac{R_s^{\,2}}{XL^2} - 4\left(1 - \frac{R_sR_l}{XL^2} + \frac{R_s^{\,2}}{XL^2}\right) = 4\left(\frac{R_sR_l}{XL^2} - 1\right)$$

The roots are:

$$Q_1 = \frac{1}{2}\left(2\frac{R_s}{XL} \pm \sqrt{\Delta}\right) = \frac{R_s}{XL} \pm \sqrt{\frac{R_sR_l}{XL^2} - 1} = \frac{R_s \pm \sqrt{R_sR_l - XL^2}}{XL}$$

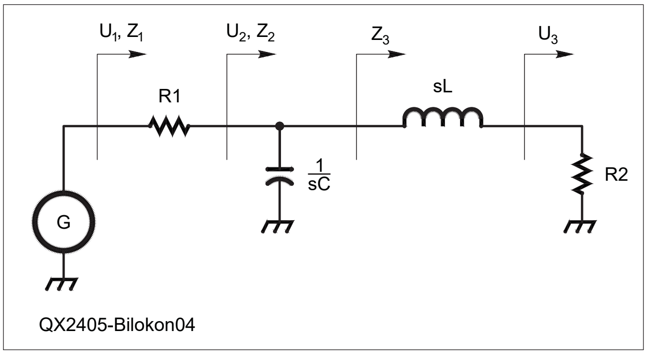


Figure 4 — The desired transfer function is equal to the ratio of the output voltage U3 to the input voltage U1 of the circuit.

Appendix 2. Derivation of the L-network harmonic suppression formula

Let's derive the transfer function for L-network. Introduce notations shown in **Figure 4**. Convert the element values into the Laplace domain. The desired transfer function is equal to the ratio of the output voltage to the input voltage of the circuit,

$$H(s) = \frac{U_3}{U_1}\ .$$

Applying Ohm's law, we can write the values of voltages and impedances at the nodes of the circuit

$$Z_2 = Z_3 \,||\, \frac{1}{sC} = \frac{1}{\frac{1}{sL+R_2} + sC} = \frac{sL + R_2}{1 + s^2LC + sCR_2}$$

$$Z_1 = R_1 + Z_2 = R_1 + \frac{sL+R_2}{1+s^2LC+sCR_2} = \frac{R_1 + s^2LCR_1 + sCR_1R_2 + sL + R_2}{1+s^2LC+sCR_2}$$

Expressing the voltage at the output of the circuit (across R_2) through the voltage of the signal source, we get

$$U_3 = U_1\frac{Z_2}{Z_1}\frac{R_2}{Z_3} = U_1\frac{R_2}{sL+R_2}\frac{sL+R_2}{1+s^2LC+sCR_2}\frac{1+s^2LC+sCR_2}{R_1+s^2LCR_1+sCR_1R_2+sL+R_2}$$

$$U_3 = U_1\frac{R_2}{R_1+s^2LCR_1+sCR_1R_2+sL+R_2}$$

The transfer function is equal to the ratio of the output voltage to the input voltage of the circuit,

$$H(s) = \frac{U_3}{U_1} = \frac{R_2}{R_1+s^2LCR_1+sCR_1R_2+sL+R_2}$$

$$= \frac{1}{\frac{R_1}{R_2} + \frac{s^2LCR_1}{R_2} + sCR_1 + \frac{sL}{R_2} + 1}$$

Let's express the variable s in terms of the harmonic number n of the fundamental frequency f_0 for which the **L**-network is designed (here i is the imaginary unit),

$$s = i\omega = i2\pi n f_0,\ n \ge 1$$

Substitute s,

$$H(n) = \frac{1}{\frac{R_1}{R_2} - \frac{(2\pi n f_0)^2 LCR_1}{R_2} + i2\pi n f_0 CR_1 + \frac{i2\pi n f_0 L}{R_2} + 1}$$

Using the basic ratios in **L**-network

$$\frac{R_1}{R_2} = Q^2 + 1,\ 2\pi f_0 C = \frac{Q}{R_1},\ 2\pi f_0 L = QR_2$$

Substituting them, we express the transfer function in terms of Q and the harmonic number n.

$$H(n) = \frac{1}{Q^2 - n^2Q^2 + 2 + i2nQ}$$

The transfer function has a complex value. Therefore, we take its magnitude,

$$|H(n)| = \frac{1}{\sqrt{(Q^2 - n^2Q^2 + 2)^2 + 4n^2Q^2}}$$

The suppression of the n-th harmonic relative to the first (i.e., the fundamental frequency on which the **L**-network is calculated) is obtained as the ratio of the powers of the first and the n-th harmonics

$$Att(n) = \frac{P_1}{P_n} = \frac{U_{3,n=1}{}^2}{R_2} \div \frac{U_3{}^2}{R_2} = \frac{|H(1)|^2}{|H(n)|^2} = \frac{(Q^2 - n^2Q^2 + 2)^2 + 4n^2Q^2}{4(Q^2+1)}$$

$$= \frac{Q^4 - n^2Q^4 + 2Q^2 - n^2Q^4 + n^4Q^4 - 2n^2Q^2 + 2Q^2 - 2n^2Q^2 + 4 + 4n^2Q^2}{4(Q^2+1)}$$

$$= \frac{Q^4 - 2n^2Q^4 + n^4Q^4 + 4Q^2 + 4}{4(Q^2+1)} = 1 + \frac{Q^4\left(n^4 - 2n^2 + 1\right)}{4(Q^2+1)}$$

$$= 1 + \frac{Q^4\left(n^2 - 1\right)^2}{4(Q^2+1)}$$

This formula describes the attenuation of **L**-network on the n-th harmonic of the fundamental frequency. Substituting the corresponding values for n makes it easy to obtain formulas for the suppression of the second and third harmonics. Since this is a ratio of powers, to convert to dB, we use the decimal logarithm,

$$A_2 = 10\log_{10}\left(1 + \frac{9Q^4}{4(Q^2+1)}\right)\ \text{dB}$$

$$A_3 = 10\log_{10}\left(1 + \frac{64Q^4}{4(Q^2+1)}\right)\ \text{dB}$$

Similar short formulas can be obtained for other harmonics.

Andrii Bilokon, UR5FFR, became interested in amateur radio at a young age thanks to his father, who taught him and helped him put together simple circuits. Andrii would like to thank Nikolai Lavrek, UXØFF, who in the 80s headed the Youth Technical Club station UB4FYC (now UX6FZZ). Thanks to Nikolai, Andrii and other youngsters were able to work on the air and participate in competitions. After a break of many years he returned to the hobby. In 2012 he received the call sign UR5FFR and for the last 10 years has been actively involved in the design of amateur radio equipment.

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